Interpreting the impact of explanatory variables in compositional models

J. Morais[1,2], C. Thomas-Agnan[1], and M. Simioni[3]
[1]Toulouse School of Economics, University of Toulouse 1 Capitole, France; *joanna.morais@live.fr*
[2]BVA, Boulogne-Billancourt, France
[3]INRA UMR 1110 MOISA, Montpellier, France

Abstract

Regression models have been developed for the case where the dependent variable is a vector of shares. Some of them, from the marketing literature, are easy to interpret but they are quite simple and can only be complexified at the expense of a large number of parameters. Other models, compositional regression models, are based on the simplicial geometry and use a log-ratio transformation of shares. They are flexible in terms of explanatory variables, but their interpretation is not straightforward, due to the link between shares. This paper combines both literatures in order to obtain a performing market-share model allowing to get relevant interpretations, which can be used for decision making in practical cases.

For example, we are interested in modeling the impact of media investments on automobile manufacturers sales. In order to take into account the competition, we model the brands market-shares as a function of brands media investments. We furthermore focus on compositional models where some explanatory variables are compositional. Two specifications are possible: in Model A, a unique coefficient is associated to each compositional explanatory variable, whereas in Model B a compositional explanatory variable is associated to component-specific and cross-effect coefficients.

Model A and Model B are estimated for our application in the B segment of the French automobile market, from 2003 to 2015. In order to enhance the interpretability of these models, we present different impact measures (marginal effects, elasticities, odds ratios) and we show that elasticities are particularly useful to isolate the impact of an explanatory variable on a particular share. We prove that elasticities can be equivalently computed from the transformed model and from the initial model. Direct and cross effects of media investments are computed for both models. Model B shows interesting non-symmetric synergies between brands.

Key words: Elasticity, odds ratio, marginal effect, compositional model, market-shares model, media investments impact.

ISBN: 978-84-947240-0-8

1 Introduction

We are interested in modeling the impact of media investments on automobile manufacturer sales. We consider that the sales volume in a particular segment of the automobile market is mainly determined by the demand through the socio-economic and regulatory context. Thus, each brand tries to have "the largest share of the cake" using marketing tools, like price and media investments. The impact of media investments of brand j on its own sales cannot be assessed without taking into account the competition. Thus, we want to model the impact of media investments on market-shares, taking into account the marketing actions of competitors, directly (cross-effects) of indirectly.

In the existing literature, we found different types of models to model shares (Morais et al. (2016) for a comparison). Some of them, from the marketing or econometric literature, are perfectly adapted to model market-shares and to interpret direct and cross impacts of media investments, but the proposed models are quite simple or can only be complexified at the expense of a very large number of parameters. Other models adapted to share data are proposed, which are called compositional regression models and are based on the simplicial geometry. These mathematical models are very flexible in terms of explanatory variables and complexity (alternative-specific and cross-effect parameters), but their interpretation is not straightforward. This paper combines both literatures in order to obtain a performing market-share model allowing to get relevant and appropriate interpretations, which can be used for example to help decision making of automobile manufacturers concerning their media investments.

Here we focus on compositional models which are coming from the so called Compositional Data Analysis (CODA) literature (Pawlowsky-Glahn et al. (2015)). A composition of D components is a vector of D shares, lying in a space called the simplex, and then respecting the following constraints: components are positive and summing up to one. Compositional models are "transformation" models in the sense that they use a log-ratio transformation of shares. Transformation models have several advantages compared to other share models: they are easy to estimate (usually by OLS on coordinates) and flexible in terms of explanatory variables (they can be compositional or classical variables, with or without component-specific parameters). More specifically, we focus on models where a compositional dependent variable is explained by some compositional explanatory variables. We make a difference between two specifications of this model: in Model A, a unique coefficient is associated to each compositional explanatory variable (Wang et al. (2013)), whereas in Model B a compositional explanatory variable is associated to component-specific and cross-effect coefficients (Chen et al. (2016)).

In compositional models, the interpretation of parameters is not straightforward as all shares are linked by the summing up to one constraint. They are usually interpreted in terms of marginal effects on the transformed shares. In this paper we propose several interpretations directly linked to the shares, in terms of marginal effects, elasticities and odds ratios, in order to enhance the interpretability of these models. We show that marginal effects on shares are not well adapted to interpret these models because they depend a lot on the considered observation. Elasticities are useful to isolate the impact of an explanatory variable on a particular share as they correspond to the relative variation of a component to the relative variation of an explanatory variable, ceteris paribus. We show that they can be computed from the transformed model or equivalently from the model in the simplex. Other types of elasticities and odds ratios can be computed for ratios of shares, which are observation independent but they can be complicated to use in practice.

Model A and Model B are applied to an automobile market data set, where the aim is to explain the brands market-shares in a segment with brands media investments. The two models are interpreted using marginal effects, elasticities and odds ratios, and they are compared in terms of (out-of-sample) goodness-of-fit using quality measures adapted for share data.

This paper is organized as follows: the second section presents the two types of compositional models; the third section explains how to interpret them; the fourth section presents the results

ISBN: 978-84-947240-0-8

of the estimation of the models for the French automobile market along with interpretations and quality measures. Finally, the last section concludes on the findings and on further directions to be investigated.

2 Compositional regression models

2.1 Definition and notations

By definition shares are compositional data: a composition is a vector of D parts of some whole which carries relative information. D-compositions lie in a space called the simplex $\mathcal{S}^D$.

$$\mathcal{S}^D = \left\{ \mathbf{s} = (s_1, s_2, \dots, s_D)' : s_j > 0,\ j = 1, \dots, D;\ \sum_{j=1}^{D} s_j = 1 \right\}$$

Compositions are subject to the following constraints: the components are positive and sum up to 1. Because of these constraints, classical regression models cannot be used directly.

The following operations are used in the simplex (Van Den Boogaart and Tolosana-Delgado (2013) for example):

- $\mathcal{C}()$ denotes the closure operation which transforms volumes into shares: $\mathcal{C}(\check{x}_1, \dots, \check{x}_D)' = \left(\frac{\check{x}_1}{\sum_{j=1}^{D} \check{x}_j}, \dots, \frac{\check{x}_D}{\sum_{j=1}^{D} \check{x}_j} \right)' = (x_1, \dots, x_D)'$ where $\check{x}$ denotes the volume and x denotes the share of a variable.
- $\oplus$ is the *perturbation operation*, corresponding to the addition operation in the simplex: $\mathbf{x} \oplus \mathbf{y} = \mathcal{C}(x_1 y_1, \dots, x_D y_D)' \qquad \text{with } \mathbf{x}, \mathbf{y} \in \mathcal{S}^D$
- $\odot$ is the *power transformation*, corresponding to the multiplication operation in the simplex: $\lambda \odot \mathbf{x} = \mathcal{C}(x_1^{\lambda}, \dots, x_D^{\lambda})' \qquad \text{with } \lambda \in \mathbb{R}, \mathbf{x} \in \mathcal{S}^D$
- $\boxdot$ is the *compositional matrix product*, corresponding to the matrix product in the simplex: $\mathbf{B} \boxdot \mathbf{x} = \mathcal{C}\big(\prod_{j=1}^{D} x_j^{b_{1j}}, \dots, \prod_{j=1}^{D} x_j^{b_{Dj}} \big)' \qquad \text{with } \mathbf{B} \in \mathbb{R}_{D \times D}, \mathbf{x} \in \mathcal{S}^D$

2.2 Log-transformation approach

Compositional data analysis is based on the log-ratio transformation of compositions in order to obtain coordinates which can be represented in a $\mathbb{R}^{D-1}$ Euclidean space[1]. Then, classical methods suited for data in the Euclidean space, like linear regression models, can be used on coordinates. Below, terms with a "*" refer to transformed elements (in coordinates), whereas terms without "*" refer to elements in the simplex (compositions).

Several transformations are developed in the CODA literature (Pawlowsky-Glahn et al. (2015) for example). The ILR (isometric log-ratio) transformation is preferred for compositional regression models. It consists in a projection of components on an orthonormal basis of $\mathcal{S}^D$ in order to obtain $D-1$ orthonormal coordinates. Considering the transformation matrix $\mathbf{V}_{D \times (D-1)}$, ILR coordinates are defined as:

$$ilr(\mathbf{s}) = \mathbf{V}' \log(\mathbf{s}) = \mathbf{s}^* = (s_1^*, \dots, s_{D-1}^*)'$$

Its inverse transformation is given by: $\mathbf{S} = ilr^{-1}(\mathbf{S}^*) = \mathcal{C}(\exp(\mathbf{V}\mathbf{S}^*))'$.

[1] Or in $\mathbb{R}^D$ in the case of the CLR transformation.

ISBN: 978-84-947240-0-8

After inverse transformation, results of a compositional analysis are the same regardless of the chosen transformation. However, ILR is preferred for compositional regression models.

2.3 Two types of compositional models

In this section, we consider two types of models adapted to a compositional dependent variable explained by compositional explanatory variable (and potentially classical variables). The difference between the two models is about the specification of the relationship between compositional explanatory and dependent variables: in contrast with Model B, Model A does not allow for component-specific and cross effect parameters associated to a compositional explanatory variable $\mathbf{X}$. In this paper, we add the possibility to use classical variables Z as explanatory variables. There is no difference between Models A and B with regard to classical variables: component-specific parameters are specified. For simplicity, models are presented with a single explanatory variable of each type (compositional X and classical Z), but of course several ones can be used like in the examples presented in Section 4.

2.3.1 Model A: Compositional dependent and explanatory variables without component-specific and cross-effect parameters

Model A is presented by Wang et al. (2013). In Model A, a compositional explanatory variable is associated to a unique parameter $b \in \mathbb{R}$ (Table 1, Eq. (1)). Thus, cross-effects[2] are not modeled directly, but indirectly through the shares closure. Indeed, we show in Morais et al. (2016) that Model A in Equation (1) can be written in attraction form like in Equation (3). This equation contains a closure, and we can see that a change of X_l will have an indirect impact on S_j through the denominator. Moreover, the attraction form of Model A enables to see that Model A respects the IIA (independence from irrelevant alternative) property. This property means that the ratio of shares of two alternatives j and l, S_j/S_l, does not depend on characteristics of other alternatives $m \neq j, l$. Note that Equation (3) can be expressed either in terms of shares X_j or in terms of volumes $\check{X}_j$ thanks to the closure operation. If a classical explanatory variable Z is used in Model A, it is associated to a composition of parameters $\mathbf{c}$[3].

The ILR transformation is used in order to estimate Model A [Eq. (5)]. Assuming that the transformed error terms are normal (implying that the non-transformed compositional error terms are "normal in the simplex"), we can use OLS to estimate the model.

An important feature of Model A is that compositional explanatory variables $\mathbf{X}$ have to be of the same dimension that the compositional dependent variable $\mathbf{S}$, such that $\mathbf{S}, \mathbf{X} \in \mathcal{S}^D$. This model is adapted when compositions $\mathbf{X}$ and $\mathbf{S}$ refer to two variables associated to the same components in the same order, for example $\mathbf{S}$ can be the composition of brands market-shares and $\mathbf{X}$ the composition of brand media investments (where brands are in the same order in $\mathbf{S}$ and $\mathbf{X}$) (see Section 4), or $\mathbf{S}$ can be the composition of GDP from three sectors and $\mathbf{X}$ the composition of labor force of these three sectors. Otherwise, this model makes no sense. Then, Equation (5) is estimated using $(D-1) \times T$ observations (the number of ILR coordinates $D-1$ times the number of observations T). Actually, this model specification is close to the specification of multinomial or market-share models (see Morais et al. (2016) for a comparison).

[2] We denote by cross-effect the effect of a variation of X_l on S_j, where $l \neq j$.

[3] It can be surprising to see that in the attraction form of Model A, the variable Z is powering the intercept c_j, but this corresponds to the term $Z_t \odot \mathbf{c}$.

ISBN: 978-84-947240-0-8

2.3.2 Model B: Compositional dependent and explanatory variables with component-specific and cross-effect parameters

Model B is used by Van Den Boogaart and Tolosana-Delgado (2013) and Chen et al. (2016) for example. Using exactly the same dependent and explanatory variables as Model A [Eq. (2)], it allows each component X_l of $\mathbf{X}$ to have a specific impact on each component S_j of $\mathbf{S}$. This is particularly visible in the attraction form of Model B [Eq. (4)]: instead of having a unique parameter $b \in \mathbb{R}$ associated to $\mathbf{X}$, we have a matrix of parameters $\mathbf{B} \in \mathbb{R}_{D_S,D_X}$. If $D_S = D_X$ and $\mathbf{S}$ and $\mathbf{X}$ refer to the same components in the same order, then $\mathbf{B}$ is a square matrix with direct effect on the diagonal and cross-effects outside of the diagonal. There is no difference between Model A and Model B for the specification of the intercept and classical explanatory variables. The same remark than for Model A can be done concerning the attraction form of Model B: Equation (4) can be expressed either in terms of shares X_j or in terms of volumes $\check{X}_j$ thanks to the closure operation.

As in Model A, in order to estimate Model B, we transform it using the ILR transformation [Eq. (6)]. But here, $D_S - 1$ equations are estimated separately (one for each coordinate of $\mathbf{S}$) with T observations each. The complexity of Model B is reflected by a large number of parameters. This can be an issue if the number of observations T is too small.

Note that in Model B, $\mathbf{X} \in \mathcal{S}^{D_X}$ and $\mathbf{S} \in \mathcal{S}^{D_S}$ can have different dimensions. For example, $\mathbf{S}$ can be the composition of GDP from three sectors and $\mathbf{X}$ the composition of labor force for six occupation categories. In our application, $D_S = D_X$: $\mathbf{S}$ is the composition of brands market-shares and $\mathbf{X}$ is the composition of brand media investments (see Section 4).

One can show that Model A is a particular case of Model B where $D_S = D_X$ and where B^* is a diagonal matrix with $b^* = b$ on the diagonal and 0 otherwise, that is where only the j^{th} ILR coordinates of compositional explanatory variables are relevant to explain the j^{th} ILR coordinates of the dependent variable (see the Appendix A.1 for demonstration in the case of $D = 3$).

Table 1: Two kinds of models for compositional dependent and explanatory variables

	Model A	Model B
In compositions	$\mathbf{S}_t = \mathbf{a} \oplus b \odot \mathbf{X}_t \oplus Z_t \odot \mathbf{c} \oplus \epsilon$ (1)	$\mathbf{S}_t = \mathbf{a} \oplus \mathbf{B} \boxdot \mathbf{X}_t \oplus Z_t \odot \mathbf{c} \oplus \epsilon$ (2)
In attraction form	$S_{jt} = \dfrac{a_j X_{jt}^b c_j^{Z_t} \epsilon_{jt}}{\sum_{m=1}^D a_m X_{mt}^b c_m^{Z_t} \epsilon_{mt}}$ (3)	$S_{jt} = \dfrac{a_j \prod_{l=1}^D X_{lt}^{b_{jl}} c_j^{Z_t} \epsilon_{jt}}{\sum_{m=1}^D a_m \prod_{l=1}^D X_{lt}^{b_{ml}} c_m^{Z_t} \epsilon_{mt}}$ (4)
In coordinates	$\mathbf{S}_t^* = \mathbf{a}^* + \mathbf{X}_t^* \cdot b + \mathbf{c}^* Z_t + \epsilon_t^*$ (5)	$\mathbf{S}_t^* = \mathbf{a}^* + \mathbf{X}_t^* \cdot \mathbf{B}_\mathbf{k}^* + \mathbf{c}^* Z_t + \epsilon_t^*$ (6)
Component-specific parameters for X	No	Yes
Cross-effects for X	No	Yes
Dimension	D for $\mathbf{S}$ and $\mathbf{X}$	D_S for $\mathbf{S}$; D_X for $\mathbf{X}$
Nb. parameters	$(D-1)(1+K_Z)+K_X$	$(D_S-1)(1+K_Z+\sum_{k=1}^{K_X}(D_k-1))$

$\mathbf{X}_t$: compositional explanatory variable; Z_t: classical explanatory variable.
D_S: number of components of $\mathbf{S}$; D_X or D_k: number of components of $\mathbf{X_k}$.
$\mathbf{S}, \mathbf{a}, \mathbf{b}, \mathbf{X}, \epsilon \in \mathcal{S}^D$; $b, X \in \mathbb{R}$; $\mathbf{B} \in \mathbb{R}^{D_S \times D_X}$; $\mathbf{S}^*, \mathbf{a}^*, \mathbf{b}^*, \mathbf{B}^*, \mathbf{X}^*, \epsilon^*$: ILR coordinates.
ϵ: normal in the simplex distributed error terms ; ϵ^*: normal distributed error terms.
K_X and K_Z: number of compositional and classical explanatory variables ($K_X = K_Z = 1$ in the table).
$\mathbb{E}^\oplus$: expected value in the simplex.

ISBN: 978-84-947240-0-8

3 Interpretation of compositional models

As the estimation of compositional models is performed in the coordinate space, the interpretation of the fitted parameters is difficult because parameters are linked to the log-ratio transformation of shares, not directly to the shares. It is possible to derive the coefficients in the simplex associated to shares using the inverse transformation, but their interpretation is not straightforward either.

We are going to show that relative impacts, like elasticities or odds ratios, are more natural (as is the case of the classical logistic model) than marginal effects, to interpret impacts on shares.

Table 2 compares the different measures of impact assessment of explanatory variables (compositional and classical) in Model A and Model B, which are detailed below. Note that it is not possible to measure the impact of the share of X_{lt}, but only of the corresponding volume of $\check{X}_{lt}$. Indeed, a share cannot increase *ceteris paribus* because it implies a change in other shares. However, we can consider a change in the volume of $\check{X}_{lt}$, with all other volumes $\check{X}_{mt}, m \neq l$ fixed.

3.1 Marginal effect of a component

In classical linear models, coefficients are usually interpreted in terms of marginal effects: if the explanatory variable increases by one, then the dependent variable increases by the value of the coefficient. In the case of compositional models, we prove in this paper that it is possible to compute marginal effects, but it is not straightforward. The marginal effect of the component $\check{X}_{lt}$ (in volume) on the dependent share S_{jt} is defined as:

$$me(\mathbb{E}^{\oplus}S_{jt}, \check{X}_{lt}) = \frac{\partial \mathbb{E}^{\oplus}S_{jt}}{\partial \check{X}_{lt}} \tag{7}$$

where $\mathbb{E}^{\oplus}S_{jt}$ is the "expected value in the simplex" of S_{jt} (Morais et al. (2016)) , such that $\mathbb{E}^{\oplus}S_{jt} = \frac{a_j X_{jt}^b c_j^{Z_t}}{\sum_{m=1}^D a_m X_{jt}^b c_m^{Z_t}}$ for Model A and $\mathbb{E}^{\oplus}S_{jt} = \frac{a_j \prod_{l=1}^D X_{lt}^{b_{jl}} c_j^{Z_t}}{\sum_{m=1}^D a_m \prod_{l=1}^D X_{lt}^{b_{ml}} c_m^{Z_t}}$ for Model B.

For Model B, we show that marginal effects can be computed as follows:

$$me(\mathbb{E}^{\oplus}S_{jt}, \check{X}_{lt}) = \frac{\partial \mathbb{E}^{\oplus}S_{jt}}{\partial \log \mathbb{E}^{\oplus}S_{jt}} \frac{\partial \log \mathbb{E}^{\oplus}S_{jt}}{\partial \log \check{X}_{lt}} \frac{\partial \log \check{X}_{lt}}{\partial \check{X}_{lt}} = \left(b_{jl} - \sum_{m=1}^{D} S_{mt} b_{ml} \right) \frac{\mathbb{E}^{\oplus}S_{jt}}{\check{X}_{lt}} \tag{8}$$

If ME_{D_S,D_X} is the matrix containing all marginal effects, we then have:

$$ME(\mathbb{E}^{\oplus}\mathbf{S}_t, \check{\mathbf{X}}_t) = [\mathbf{S_{jt}}]\, \mathbf{W}_t \mathbf{B} \odot \left[\frac{\mathbf{1}}{\check{\mathbf{X}}_{\mathbf{lt}}}\right] = [\mathbf{S_{jt}}] \odot \mathbf{W}_t \mathbf{V} \mathbf{B}^* \mathbf{V}' \odot \left[\frac{\mathbf{1}}{\check{\mathbf{X}}_{\mathbf{lt}}}\right] \tag{9}$$

where $\odot$ denotes the Hadamard product here (term by term product)[4], $[\mathbf{S_{jt}}]$ is a $D_S \times D_S$ matrix with S_{jt} on the j^{th} row, $\left[\frac{\mathbf{1}}{\check{\mathbf{X}}_{\mathbf{lt}}}\right]$ is a $D_X \times D_X$ matrix with $\check{X}_{lt}$ on the l^{th} column, $\mathbf{B}^*$ and $\mathbf{B}$ denote the parameters in the transformed space and in the simplex, and $\mathbf{W}_t$ is a $D_S \times D_S$ matrix composed of diagonal terms equal to $1 - \mathbb{E}^{\oplus}S_j$ and non-diagonal terms in column j equal to $-\mathbb{E}^{\oplus}S_j$. Similar results can be found for Model A in Table 2, where $\mathbf{B}$ is replaced by b.

This marginal effect matrix can also be computed using ILR coordinates and Jacobian matrices instead of using the attraction form of the model (Appendix A.2).

3.2 Elasticity of a dependent share relative to a component

The marginal effect $me(\mathbb{E}^{\oplus}S_{jt}, \check{X}_{lt})$ depends on all shares S_{mt} and on volumes $\check{X}_{lt}$. Thus, it can vary a lot across observations, and therefore it is not a good measure to summarize the impact of a

[4]Note that $\odot$ in bold denotes the Hadamard product whereas $\odot$ denotes the power transformation.

ISBN: 978-84-947240-0-8

component $\check{X}_{lt}$ on a share S_{jt}. We are going to show that elasticities are more natural to interpret compositional models.

The first elasticity we may want to compute is the elasticity of the share S_{jt} relative to the volume of $\check{X}_{lt}$. It corresponds to the relative variation of S_{jt} induced by a relative variation of 1% of $\check{X}_{lt}$:

$$e_{jlt} = e(\mathbb{E}^{\oplus} S_{jt}, \check{X}_{lt}) = \frac{\frac{\partial \mathbb{E}^{\oplus} S_{jt}}{\mathbb{E}^{\oplus} S_{jt}}}{\frac{\partial \check{X}_{lt}}{\check{X}_{lt}}} = \frac{\partial \log \mathbb{E}^{\oplus} S_{jt}}{\partial \log \check{X}_{lt}} \tag{10}$$

These elasticities are easy to compute from the attraction form of $\mathbb{E}^{\oplus} S_{jt}$, in a similar way than marginal effects [Eq. (8)]. They can also be expressed in a matrix form $E(\mathbb{E}^{\oplus}\mathbf{S}_t, \check{\mathbf{X}}_t)$ (results are in Table 2). The relationship between marginal effects and elasticities is as follows:

$$ME(\mathbb{E}^{\oplus}\mathbf{S}_t, \check{\mathbf{X}}_t) = [\mathbf{S_{jt}}] \odot E(\mathbb{E}^{\oplus}\mathbf{S}_t, \check{\mathbf{X}}_t) \odot \left[\mathbf{1}/\check{\mathbf{X}}_{\mathbf{lt}}\right]$$

These elasticities allow to isolate the impact of one $\check{X}$'s component on one S's component which is very useful. $e(\mathbb{E}^{\oplus} S_{jt}, \check{X}_{lt})$ depends on observations but only through the S_{mt}, not through $\check{X}_{lt}$. Then, if shares are not varying to much, as it is the case in our example (see Section 4), they can be a good measure of impact.

As for marginal effects, the elasticity matrix can also be computed from ILR coordinates (Appendix A.2).

Note that for a small relative change of $\check{X}_{lt}$ equal to $h = \frac{\Delta \check{X}_{lt}}{\check{X}_{lt}}$, a first order Taylor approximation of the share denoted S'_{jt} is:

$$S'_{jt} = S_{jt}(1 + h e_{jlt}) \tag{11}$$

We can verify that, for a small h, the S'_{mt} do belong to the simplex (they are summing up to one because $\sum_{m=1}^{D} \mathbb{E}^{\oplus} S_{mt} e_{jlt} = 0$, see proof in the Appendix A.3).

Moreover, we can link these elasticities to simplicial derivatives[5] (i.e. derivatives in the simplex). Indeed, the simplicial derivative of the composition $\mathbf{S}$ with respect to the log of a particular component $\check{X}_l$ is defined as follows:

$$e^{\oplus}_{lt} = \frac{\partial^{\oplus} \mathbb{E}^{\oplus} \mathbf{S}_t}{\partial^{\oplus} \log \check{X}_{lt}} = \mathcal{C}\left(\exp\left(\frac{\partial \log \mathbb{E}^{\oplus} \mathbf{S}_t}{\partial \log \check{X}_{lt}}\right)\right) = \mathcal{C}\left(\exp(e_{1lt}), \ldots, \exp(e_{Dlt})\right) \tag{12}$$

For a small relative change of $\check{X}_l$ equal to $h = \frac{\Delta \check{X}_{lt}}{\check{X}_{lt}}$, another first order Taylor approximation of share denoted $\mathbf{S}''_t$ is[6]:

$$\mathbf{S}''_t = \mathbf{S}_t \oplus h \odot e^{\oplus}_{lt} = \mathcal{C}\left(S_{1t} \exp(h e_{1lt}), \ldots, S_{Dt} \exp(h e_{Dlt})\right) \tag{13}$$

Note that when $h \to 0$, $\exp(h e_{jlt}) \simeq 1 + h e_{jlt}$, so that:

$$\mathbf{S}''_t \simeq \mathcal{C}\left(S_{1t}(1 + h e_{1lt}), \ldots, S_{Dt}(1 + h e_{Dlt})\right) = \mathcal{C}\left(S'_{1t}, \ldots, S'_{Dt}\right) = (S'_{1t}, \ldots, S'_{Dt}) \tag{14}$$

where S'_{jt} are computed in Equation (11) and $\mathbf{S}''_t$ in Equation (13). The last equality of Equation (14) is justified by the fact that $\sum_{m=1}^{D} \mathbb{E}^{\oplus} S_{mt} = 1$ and $\sum_{m=1}^{D} \mathbb{E}^{\oplus} S_{mt} e_{jlt} = 0$.

3.3 Elasticity and odds ratio of a ratio of dependent shares relative to a component

In order to avoid being observation dependent, other measures can be computed for interpreting Models A and B. However, they are concerning ratios of shares, not directly a single share. Then, they can be complicated to interpret in practical cases.

[5] See Equation (9.9), p.183, in Pawlowsky-Glahn et al. (2015).

[6] See Equation (12.13), p.168, in Pawlowsky-Glahn and Buccianti (2011).

ISBN: 978-84-947240-0-8

Elasticity of a ratio of dependent shares As compositional data analysis is based on a log ratio approach, elasticities of ratios are easy to compute. We can be interested in the elasticity of a ratio of shares (or volumes) $\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}$ relative to an infinitesimal change in the volume of $\check{X}_{lt}$.

$$e(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}, \check{X}_{lt}) = \frac{\partial \log(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't})}{\partial \log \check{X}_{lt}} \tag{15}$$

We see in Table 2 that the result is constant across observations because it only depends on parameters. Note here that Model A respects the IIA (Independence from Irrelevant Alternatives) property, meaning that the ratio of two shares $\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}$ only depends on the corresponding components j and j' of $\check{\mathbf{X}}$. Then, $e(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}, \check{X}_{lt}) = 0$ if $l \neq j, j'$. Moreover, the elasticity of the ratio between the share j and the share j' relative to a change in $\check{X}_{jt}$ is the same for all considered shares j'. This is a lack of flexibility of Model A, because it implies that an increase of $\check{X}_{jt}$ will reduce proportionally all other shares. Model B does not satisfy the IIA property, and then this model is able to take into account possible synergies between brands.

Odds ratio of a ratio of dependent shares Another type of interpretation which can be used for shares is the odds ratio. The advantage of this measure is that it is a measure of impact of a discrete change, as opposed to infinitesimal change, of $\check{X}_l$ ($\check{X}_l$ is increased by $\Delta \times 100\%$ between situations $t = t1$ and $t = t2$) on the ratio $\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}$. The empirical odds ratio for a couple of shares $\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}$ relative to $\check{X}_{lt}$ is given by:

$$OR(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}, \check{X}_{lt}, \Delta) = \frac{(\mathbb{E}^{\oplus}S_{j,t2}/\mathbb{E}^{\oplus}S_{j',t2})|\check{X}_{l,t2}}{(\mathbb{E}^{\oplus}S_{j,t1}/\mathbb{E}^{\oplus}S_{j',t1})|\check{X}_{l,t1}} \tag{16}$$

where $\check{X}_{l,t2} = (1+\Delta)\check{X}_{l,t1}$ and $\Delta \geq 0$.

Remark: $e(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}, \check{X}_{lt})$ and $OR(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}, \check{X}_{lt}, \Delta)$ are more or less measuring the same thing differently, if Δ is small:

$$\begin{aligned} e(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}, \check{X}_{lt}) &\simeq \frac{(\mathbb{E}^{\oplus}S_{jt2}/\mathbb{E}^{\oplus}S_{j't2}) - (\mathbb{E}^{\oplus}S_{jt1}/\mathbb{E}^{\oplus}S_{j't1})}{(\mathbb{E}^{\oplus}S_{jt1}/\mathbb{E}^{\oplus}S_{j't1})} / \frac{\check{X}_{lt2} - \check{X}_{lt1}}{\check{X}_{lt1}} \\ &\simeq \frac{OR(\mathbb{E}^{\oplus}S_{jt}/\mathbb{E}^{\oplus}S_{j't}, \check{X}_{lt}, \Delta) - 1}{(\check{X}_{lt2} - \check{X}_{lt1})/(\check{X}_{lt1})} \end{aligned}$$

3.4 Elasticity of a particular ratio of dependent shares relative to a particular ratio of components

Usually, compositional models are interpreted directly on coordinates. Thus, it is advised to choose an appropriate ILR transformation in order to have ILR coordinates which make sense for the considered application, using sequential binary partition for example (Hron et al. (2012)). But, previously the interpretation was made in terms of marginal effects on ILR coordinates, that is marginal effects on a particular log ratio of shares. We show here that we can go a step further and make an interpretation in terms of elasticity for the ratio of shares directly.

Chen et al. (2016) interpret in the case of Model B the impact of the ratio $X_l/g(X_{-l}) = \check{X}_l/g(\check{X}_{-l})$ on the ratio $\mathbb{E}^{\oplus}S_j/g(\mathbb{E}^{\oplus}S_{-j}) = \mathbb{E}^{\oplus}\check{S}_j/g(\mathbb{E}^{\oplus}\check{S}_{-j})$ (ratios on shares or volumes are equivalent), which is the ratio of a particular share (or volume) S_j over the geometric average of other shares (or volumes). The adapted ILR transformation is the following:

$$ilr(\mathbf{X})_i = \sqrt{\frac{D-i}{D-i+1}} \log \frac{x_i}{(\prod_{j=1+i}^{D} x_j)^{1/(D-i)}}, \quad i = 1, \ldots, D-1$$

ISBN: 978-84-947240-0-8

With this transformation, the first expected coordinate of **S** in Model A, is equal to:

$$\mathbb{E}ilr(\mathbf{S})_1 = \sqrt{\frac{D-1}{D}}\log\frac{\mathbb{E}^{\oplus}S_{1t}}{g(\mathbb{E}^{\oplus}S_{-1t})} = a_1^* + b^*\sqrt{\frac{D-1}{D}}\log\frac{\check{X}_{1t}}{g(\check{X}_{-1t})} + c_1^*Z_t$$

In Model B, the first expected coordinate of **S** is equal to:

$$\mathbb{E}ilr(\mathbf{S})_1 = \sqrt{\frac{D_S-1}{D_S}}\log\frac{\mathbb{E}^{\oplus}S_{1t}}{g(\mathbb{E}^{\oplus}S_{-1t})} = a_1^* + b_{11}^{*(j,l)}\sqrt{\frac{D_X-1}{D_X}}\log\frac{\check{X}_{1t}}{g(\check{X}_{-1t})} + b_{12}^{*(j,l)}\sqrt{\frac{D_X-2}{D_X-1}}\log\frac{\check{X}_{2t}}{g(\check{X}_{-1-2t})} + \dots$$

In order to interpret their model, Chen et al. (2016) compute the marginal effect of $ilr(X)_1^{(l)}$ on $ilr(S)_1^{(j)}$:

$$me(\mathbb{E}ilr(S)_1^{(j)}, ilr(\check{X})_1^{(l)}) = \frac{\partial\sqrt{\frac{D_S-1}{D_S}}\log(\mathbb{E}^{\oplus}S_{jt}/g(\mathbb{E}^{\oplus}S_{-jt}))}{\partial\sqrt{\frac{D_X-1}{D_X}}\log(\check{X}_{lt}/g(\check{X}_{-lt}))} = b_{11}^{*(j,l)}$$

such that an increase of one unit of $ilr(\check{X})_1^{(l)}$ implies an increase of $b_{11}^{*(j,l)}$ units of $\mathbb{E}ilr(S)_1^{(j)}$[7].

Note that this is only true if $\sqrt{\frac{D_X-1}{D_X}}\log(X_{lt}/g(X_{-lt}))$ moves because $\check{X}_{1t}$ moves while other $\check{X}_{jt}$ remain constant. Otherwise, other ILR coordinates in the right part of the equation are moving and the marginal effect should take it into account. However, for Model A, we do not have this problem because other ILR coordinates of **X** are not used.

We show that this is equivalent to compute the following elasticity (multiplying by a factor if $D_S \neq D_X$):

$$e\left(\frac{\mathbb{E}^{\oplus}S_{jt}}{g(\mathbb{E}^{\oplus}S_{-jt})}, \check{X}_{lt}\right) = \frac{\partial\log(\mathbb{E}^{\oplus}S_{jt}/g(\mathbb{E}^{\oplus}S_{-jt}))}{\partial\log\check{X}_{lt}} = \sqrt{\frac{(D_X-1)/D_X}{(D_S-1)/D_S}}b_{11}^{*(j,l)}$$

Thus, instead of saying that when $ilr(\check{X})_1^{(l)}$ increases by 1 unit, $\mathbb{E}ilr(S)_1^{(j)}$ increases by $b_{11}^{*(j,l)}$ units, one can say that when $\check{X}_{lt}$ increases by 1%, $\mathbb{E}^{\oplus}S_{jt}/g(\mathbb{E}^{\oplus}S_{-jt})$ increases by $b_{11}^{*(j,l)}$% (in the case where $D_S = D_X$). Note that this $b_{11}^{*(j,l)}$ will be different for each permutation (i.e. each couple j,l). Chen et al. (2016) show how one can determine in one step the first coefficient of $B^{*(j,l)}$, the $b_{11}^{*(j,l)}$ which is used to compute the above elasticity, for all possible permutations without fitting several times the model.

3.5 Elasticities and odds ratios relative to a classical variable

The same kind of interpretations can be done for classical variables Z, as presented in Table 2, except for the elasticity including the geometrical mean.

Indeed, this would allow to measure the marginal effect (not the elasticity) of Z_t over $\sqrt{\frac{D_S-1}{D_S}}\log\frac{S_{1t}}{g(S_{-1t})}$. This marginal effect would be equal to c_1^* for Model A and Model B, but this kind of interpretation is not useful to understand the impact of Z on the final shares. Thus, we do not show this measure in Table 2.

Note that in practice, elasticities and other measures depending on $\mathbb{E}^{\oplus}S_{jt}$ are estimated using the observed shares S_{jt}, not the fitted shares $\widehat{S_{jt}}$.

[7] $ilr(S)_1^{(j)}$ denotes the first ILR coordinate of **S** where S_j is in the first position; $ilr(\check{X})_1^{(l)}$ denotes the first ILR coordinate of $\check{\boldsymbol{X}}$ where $\check{X}_l$ is in the first position.

ISBN: 978-84-947240-0-8

Table 2: Measures of impact assessment for Model A and Model B

Var	Measure	Effect	Model A	Model B
X	$me(S_{jt}, \check{X}_{lt})$	Direct	$b(1-S_{jt})\frac{S_{jt}}{\check{X}_{lt}}$	$(b_{jl} - \sum_{m=1}^{D} S_{mt} b_{ml})\frac{S_{jt}}{\check{X}_{lt}}$
		Indirect	$(-bS_{lt})\frac{S_{jt}}{\check{X}_{lt}}$	
	$ME(\mathbf{S}_t, \check{\mathbf{X}}_t)$	Matrix	$[\mathbf{S_{jt}}] \odot \mathbf{W_t} b \odot [\mathbf{1}/\check{\mathbf{X}}_{\mathbf{lt}}]$	$[\mathbf{S_{jt}}] \odot \mathbf{W_t B} \odot [\mathbf{1}/\check{\mathbf{X}}_{\mathbf{lt}}]$
	$e(S_{jt}, \check{X}_{lt})$	Direct	$b(1-S_{jt})$	$(b_{jl} - \sum_{m=1}^{D} S_{mt} b_{ml})$
		Indirect	$-bS_{lt}$	
	$E(\mathbf{S}_t, \check{\mathbf{X}}_t)$	Matrix	$\mathbf{W_t} b$	$\mathbf{W_t B}$
	$e\left(\frac{S_{jt}}{S_{j't}}, \check{X}_{lt}\right)$	Direct	b	$(b_{jl} - b_{j'l})$
		Indirect	0	
	$\mathrm{OR}\left(\frac{S_{jt}}{S_{j't}}, \check{X}_{lt}, \Delta\right)$	Direct	$(1+\Delta)^b$	$(1+\Delta)^{(b_{jl}-b_{j'l})}$
		Indirect	0	
	$e\left(\frac{S_{jt}}{g(S_{-jt})}, \check{X}_{lt}\right)$	Direct	b	$b_{11}^{*(j,l)}\sqrt{\frac{D_X-1}{D_X}}/\sqrt{\frac{D_S-1}{D_S}}$
		Indirect	0	
Z	$me(S_{jt}, Z_t)$		$(\log c_j - \sum_{m=1}^{D} S_{mt}\log c_m)S_{jt}$	
	$ME(\mathbf{S}_t, Z_t)$	Vector	$[\mathbf{S_{jt}}] \odot \mathbf{W}_t \log \mathbf{c}$	
	$e(S_{jt}, Z_t)$		$(\log c_j - \sum_{m=1}^{D} S_{mt}\log c_m)Z_t$	
	$E(\mathbf{S}_t, Z_t)$	Vector	$\mathbf{W}_t \log \mathbf{c} \cdot Z_t$	
	$e(\frac{S_{jt}}{S_{j't}}, Z_t)$		$\log(c_j/c_{j'})Z_t$	
	$\mathrm{OR}\left(\frac{S_{jt}}{S_{j't}}, Z_t, \Delta\right)$		$(c_j/c_{j'})^{\Delta Z_t}$	

In this table, $\mathbb{E}^{\oplus} S_{jt}$ is denoted by S_{jt} to shorten notations, and $\odot$ denotes the Hadamard product.
Moreover, these measures are estimated using observed shares S_{jt} in practice, not fitted shares.
Direct effect when $l = j$; indirect effect when $l \neq j$.
$\mathbf{W_t}$ contains $1 - S_{it}$ on the diagonal and $-S_{it}$ otherwise.

4 Impact of media investments on brands market-shares

In Europe, the automobile market is usually segmented in 5 segments, from A to E, according to the size of the vehicle chassis. Within each segment, one can suppose that consumers intending to buy new cars make their choice between brands[8] according to the price and the "image" of the brand. The image of the brand is supposed to reflect the notion of quality and reliability of the brand. Car manufacturers spend millions of euros in media investments to enhance their image, giving rise to the following question: do the media investments have an impact on brands market-shares[9]?
In order to answer this question in the present paper, we model brands market-shares of the B segment of the French automobile market[10] as a function of brand media investments (in TV, radio, press, outdoor, internet and cinema), of brand average catalogue price and of a scrapping incentive dummy variable. In a further work, we consider modeling other segments, and differentiate media investments according to channels.

In this paper, three brands are highlighted (Renault, Peugeot, Citroen, the leaders of the B segment) while other brands of the B segment are aggregated in a category "Others" (Fig. 1). The media investments are the sum of TV, radio, press, outdoor, internet and cinema investments in euros by brands for their vehicles in the B segment (Fig. 1). They do not include advertising budget for the brand itself. Actually we use the media investments of one, two and three months before the purchase time (at time $t-1, t-2, t-3$) as explanatory variables. The average brand price (average of catalogue prices weighted by corresponding sales at the vehicle level) is also used as an explanatory variable (Fig. 1). It does not include potential promotions made in the car

[8] Inside a segment, a brand generally supplies only one main vehicle. Thus, we can consider that the alternatives for a consumer inside a particular segment coincide with the available brands in this segment.

[9] We decide to ask the question in terms of market-shares instead of in terms of sales volumes because one can suppose that at time t, brands have to share a market for which the size is mainly determined by the demand.

[10] The B segment is the most important segment in terms of sales in France (around 40% of new passenger car sales).

ISBN: 978-84-947240-0-8

dealership at the time of purchase. Even if they do not vary a lot across time, prices are used to position brands within the segment. We also control for scrapping incentive periods. The corresponding dummy variable is a "classical" variable (not compositional) and varies across time only, not across brands.

Model A and Model B can be considered in this framework: Model A considers that the effect of media investments and price are the same for all brands whereas Model B implies cross-effects and brand-specific impacts of media investments and price on market-shares.

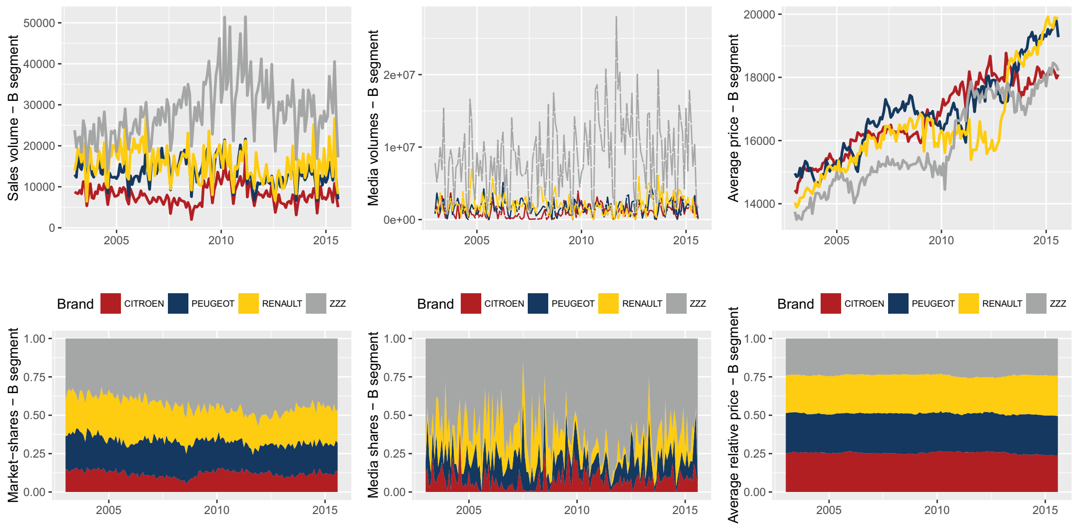

Figure 1: Sales, media and average price of brands, in volume and in share, in the B segment

This section presents the results of this application. We interpret the two models A and B in terms of elasticities and odds ratios of shares, and we compare them in terms of goodness-of-fit measures.

4.1 Non brand-specific impact of media investments (Model A)

Model In the case where it is assumed that brand media investments and brand prices have the same effect for all brands, the following equations correspond to the model in the simplex and the attraction formulation of the model:

$$\mathbf{S}_t = \mathbf{a} \bigoplus_{\tau=1}^{3} b_\tau \odot \mathbf{M}_{t-\tau} \oplus b_P \odot \mathbf{P}_t \oplus SI_t \odot \mathbf{c} \oplus \boldsymbol{\varepsilon}_t$$

$$\Leftrightarrow S_{jt} = \frac{a_j \cdot \prod_{\tau=1}^{3} M_{t-\tau,j}^{b_\tau} \cdot P_{t,j}^{b_P} \cdot c_j^{SI} \cdot \varepsilon_{jt}}{\sum_{m=1}^{4} a_m \cdot \prod_{\tau=1}^{3} M_{t-\tau,m}^{b_\tau} \cdot P_{t,m}^{b_P} \cdot c_m^{SI} \cdot \varepsilon_{mt}}$$

where $\mathbf{S}, \mathbf{M_{t-\tau}}, \mathbf{P} \in \mathcal{S}^4$ are the compositions of brand sales, of brand media investments at time $t-1$, $t-2$ and $t-3$, and of brand prices. $b_\tau, b_P \in \mathbb{R}$ are the parameters associated to compositional explanatory variables and $\mathbf{c} \in \mathcal{S}^4$ is a composition of parameters associated to the dummy variable SI (scrapping incentive).

ISBN: 978-84-947240-0-8

The ILR transformed version of the model is:

$$\mathbf{S}_t^* = \mathbf{a}^* + \sum_{\tau=1}^{3} b_\tau \mathbf{M}_{t-\tau}^* + b_P \mathbf{P}_t^* + \mathbf{c}^* SI_t + \boldsymbol{\varepsilon}_t^*$$

$$\Leftrightarrow S_{jt}^* = a_j^* + \sum_{\tau=1}^{3} b_\tau^* M_{j,t-\tau}^* + b_P^* P_{jt}^* + c_j^* SI_t + \varepsilon_{jt}^* \qquad \text{for } j = 1, 2, 3$$

where $\boldsymbol{\varepsilon}^*$ is supposed to be a Gaussian distributed error term. The balance matrix used for the ILR transformation is the default matrix in the R software:

$$V_{ILR,4} = \begin{bmatrix} -\sqrt{1/2} & -\sqrt{1/6} & -\sqrt{1/12} \\ \sqrt{1/2} & -\sqrt{1/6} & -\sqrt{1/12} \\ 0 & \sqrt{2/3} & -\sqrt{1/12} \\ 0 & 0 & \sqrt{3/4} \end{bmatrix} \qquad (17)$$

Results All explanatory variables are significant at 0.1% according to the analysis of variance (ANOVA). Figure 2 compares observed and fitted shares. It confirms that the model succeeds in fitting the main trends of brands market-shares. However, the model underestimates the market-share of "Others" at the beginning of the period, and overestimates it at the end.

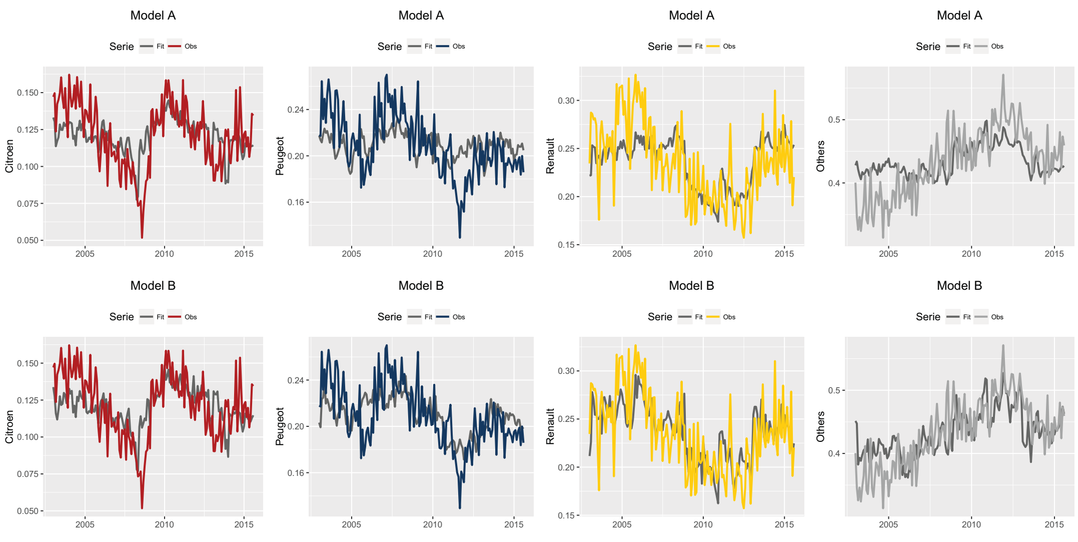

Figure 2: Observed (color) and predicted (grey) brands market-shares

The parameters estimated on the ILR transformed model are presented in Table 3. The corresponding parameters for the model in the simplex are in Table 4. We remark that the coefficient associated to the price is positive, which can be surprising, but price here is correlated with the image of quality of the brand, which is very important for the customer who buy a durable and expensive good like a car.

4.2 Brand-specific impact of media investments (Model B)

Model Now, let us look at a different specification of the model (dependent and explanatory variables are the same as in Model A) where brand-specific coefficients are assumed and cross-

ISBN: 978-84-947240-0-8

Table 3: Estimated parameters on ILR coordinates - Model A

	Estimate	Std. Error	t value	Pr(> |t|)
a_1^*	0.3439	0.0151	22.84	0.0000***
a_2^*	0.3363	0.0159	21.19	0.0000***
a_3^*	0.6620	0.0263	25.14	0.0000***
b_1	0.0267	0.0071	3.79	0.0002***
b_2	0.0241	0.0062	3.90	0.0001***
b_3	0.0264	0.0062	4.26	0.0000***
b_P	1.2217	0.2313	5.28	0.0000***
c_1^*	-0.0241	0.0338	-0.71	0.4758
c_2^*	-0.1690	0.0334	-5.05	0.0000***
c_3^*	0.1292	0.0336	3.84	0.0001***
Nb param.	10			

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Table 4: Estimated parameters in the simplex - Model A

	S_1 (Citroen)	S_2 (Peugeot)	S_3 (Renault)	S_4 (Others)
(Intercept)	0.1300	0.2114	0.2502	0.4084
M_{t-1}		0.0267		
M_{t-2}		0.0241		
M_{t-3}		0.0264		
P_t		1.2217		
SI	0.2610	0.2523	0.2086	0.2780

effects are directly modeled. It corresponds to the following model:

$$\mathbf{S}_t = \mathbf{a} \bigoplus_{\tau=1}^{3} \mathbf{B}_\tau \boxdot \mathbf{M}_{t-\tau} \oplus \mathbf{B_P} \boxdot \mathbf{P}_t \oplus SI_t \odot \mathbf{c} \oplus \boldsymbol{\varepsilon}_t$$

$$\Leftrightarrow S_{jt} = \frac{a_j \cdot \prod_{\tau=1}^{3} \prod_{l=1}^{4} M_{t-\tau,l}^{b_{\tau,jl}} \cdot \prod_{l=1}^{4} P_{t,l}^{b_{P,jl}} \cdot c_j^{SI} \cdot \varepsilon_{jt}}{\sum_{m=1}^{4} a_m \cdot \prod_{\tau=1}^{3} \prod_{l=1}^{4} M_{t-\tau,l}^{b_{\tau,ml}} \cdot \prod_{l=1}^{4} P_{t,l}^{b_{P,ml}} \cdot c_m^{SI} \cdot \varepsilon_{mt}}$$

where $\mathbf{B}_\tau, \mathbf{B_P} \in \mathbb{R}^{D\times D}$ are the matrices of parameters associated to compositional explanatory variables.

The corresponding ILR transformed model is:

$$\mathbf{S}_t^* = \mathbf{a}^* + \sum_{\tau=1}^{3} \mathbf{B}_\tau^* \mathbf{M}_{t-\tau}^* + \mathbf{B_P^*} \mathbf{P}_t^* + \mathbf{c}^* SI_t + \boldsymbol{\varepsilon}_t^*$$

$$\Leftrightarrow S_{jt}^* = a_j^* + \sum_{\tau=1}^{3} \sum_{l=1}^{3} b_{\tau,jl}^* M_{l,t-\tau}^* + \sum_{l=1}^{3} b_{P,jl}^* P_{lt}^* + c_j^* SI_t + \varepsilon_{jt}^* \quad \text{for } j = 1,2,3$$

where $\boldsymbol{\varepsilon}^*$ is supposed to be a Gaussian distributed error term. The same balance matrix $V_{ILR,4}$ is used.

Results All variables of the model are significant at 0.1% according to the ANOVA, except the price which is significant at 1%. According to Figure 2, Model B seems to fit better than Model A (see Section 4.3 for associated quality measures). The estimated parameters of the models are given in Table 5 and Table 6.

ISBN: 978-84-947240-0-8

Table 5: Estimated parameters on ILR coordinates - Model B

	S_1^* (Peu. vs Cit.)	S_2^* (Reu. vs Cit.,Peu.)	S_3^* (Oth. vs Cit.,Peu.,Reu.)
(Intercept)	0.3686***	0.3637***	0.6940***
$M^*_{t-1,1}$	0.0193.	-0.0052	0.0081
$M^*_{t-1,2}$	0.0162	0.0319*	-0.0245
$M^*_{t-1,3}$	-0.0069	0.0009	0.0279
$M^*_{t-2,1}$	0.0208.	-0.0093	0.0205.
$M^*_{t-2,2}$	0.0151	0.0361**	-0.0259.
$M^*_{t-2,3}$	-0.0197	-0.0338.	0.0278
$M^*_{t-3,1}$	0.0289**	-0.0115	0.0278*
$M^*_{t-3,2}$	0.0104	0.0206*	-0.0274.
$M^*_{t-3,3}$	-0.0114	0.0064	0.0323.
P_1^*	0.8854.	-0.5981	1.9138***
P_2^*	0.0151	0.2615	0.6509
P_3^*	-0.6442	-0.3729	2.4717***
SI^*	-0.0394	-0.2088***	0.2070***
Adjusted R2	0.3353	0.3255	0.3269
Nb param.	42		

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Table 6: Estimated parameters of M_{t-1} in the simplex - Model B

	S_1 (Citroen)	S_2 (Peugeot)	S_3 (Renault)	S_4 (Others)
$M_{t-1,1}$	0.0179	-0.0079	-0.0067	-0.0032
$M_{t-1,2}$	-0.0016	0.0111	-0.0161	0.0066
$M_{t-1,3}$	-0.0132	0.0084	0.0292	-0.0243
$M_{t-1,4}$	-0.0030	-0.0115	-0.0064	0.0209

4.3 Interpretation of models A and B

Marginal effect of media investments We calculate the marginal effects of media investments at time $t-1$ on market-shares at time t. The average marginal effects are reported in Table 7. They are quite consistent between Model A and Model B, with positive direct marginal effects and negative cross marginal effects. However, these measures are not really adapted to summarize an impact as they fluctuate a lot across time, as we can see in Figure 3 (marginal effects can be larger than 6e-08 but we voluntarily cropped the graph). The marginal effects of Citroen media investments are especially very high when these investments are very low, for example between 2007 and 2009.

Table 7: Average marginal effects of media investments $\check{M}_{t-1}$ on market-shares

$me(S_{jt}, \check{M}_{l,t-1})$	Model A				Model B			
	$\check{M}_{C,t-1}$	$\check{M}_{P,t-1}$	$\check{M}_{R,t-1}$	$\check{M}_{Z,t-1}$	$\check{M}_{C,t-1}$	$\check{M}_{P,t-1}$	$\check{M}_{R,t-1}$	$\check{M}_{Z,t-1}$
$S_{Citroen,t}$	**1.93e-05**	-1.65e-09	-2.13e-09	-3.01e-10	**1.68e-05**	-7.20e-10	-2.82e-09	-2.00e-10
$S_{Peugeot,t}$	-4.58e-06	**1.14e-08**	-3.09e-09	-5.30e-10	-7.67e-06	**5.51e-09**	7.72e-09	-7.52e-10
$S_{Renault,t}$	-4.88e-06	-3.64e-09	**1.35e-08**	-5.96e-10	-6.43e-06	-1.14e-08	**2.23e-08**	-5.71e-10
$S_{Others,t}$	-9.89e-06	-6.10e-09	-8.24e-09	**1.43e-09**	-2.66e-06	6.60e-09	-2.72e-08	**1.52e-09**

C: Citroen; P: Peugeot; R: Renault; Z: Others.

Figures in bold: direct elasticities.

Elasticity of the share S_j relative to X_l For Model A, cross elasticities are necessarily negative and direct elasticities are necessarily positive if the parameter b is positive. Moreover, cross-elasticities of market-shares S_j with respect to a particular media budget $M_{l,t-1}$ are equal

ISBN: 978-84-947240-0-8

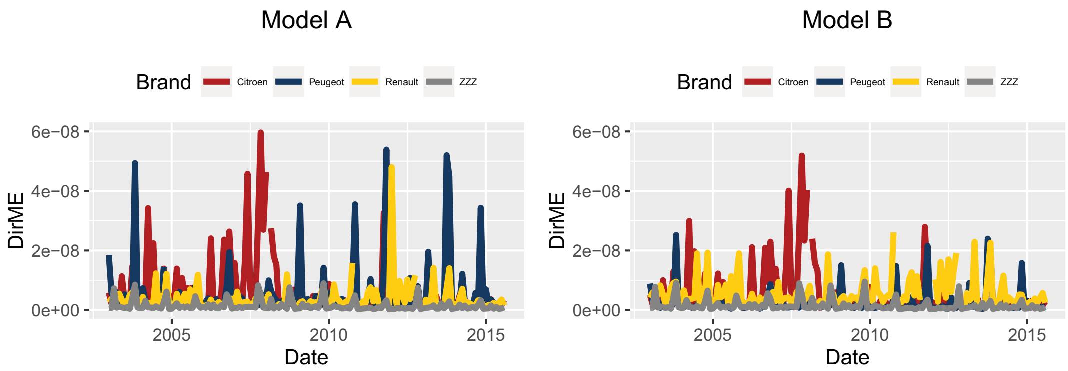

Figure 3: Direct marginal effects of $M_{j,t-1}$ on S_{jt} across time

for any brand $j \neq l$. This is a lack of flexibility of Model A compared to Model B: it does not allow positive interaction between brands, and it considers that if a brand increases its media investments of 1% it will affect in the same way all competitors market-shares S_j (they will all decrease by b%).

Let us consider a situation where the market shares of Citroen, Peugeot, Renault and Others in the B segment are respectively 10%, 25%, 25% and 40%. According to Table 8, if Renault increases its media investments M_{t-1} about 1%, the average elasticity of Model A on the studied period suggests that its market-share should increase by 0.0204% to reach 25.005% and that competitors market-shares should decrease by 0.0204% to reach respectively 9.998%, 24.995% and 39.992%[11].

In Model B, when brand-specific effects and cross-effects are taken into account, the direct elasticity of Renault market-share in the B segment relative to its corresponding media investments is much higher than other brands (0.0327), contrary to Peugeot which has the lowest (0.0099). Note that positive cross-effects (synergies) are possible in Model B: for example when Renault invests more in media, it tends to help its own market-share a lot, but also to raise a little bit the share of Peugeot, and to have a negative impact on Citroen and Others. Then, after closure and depending on the considered values of S_j, an increase in Renault media investments in the B segment can increase or decrease the Peugeot market-share.

Taking the same example as previously, according to Model B, if Renault increases its media investments M_{t-1} of about 1%, the average elasticity on the studied period suggests that its market-share should increase by 0.0327% to reach 25.008% and that competitors market-shares should respectively decrease by 0.0097%, increase by 0.0119% and decrease by 0.0208% to reach respectively 9.999%, 25.003% and 39.992.

As shown in Figure 4, the estimated direct elasticities are quite stable across time. However, as elasticities in Model A are computed using the same parameter b for all brands, they are closer to each other than in Model B where they are computed using different parameters b_{jl}. The direct elasticity of Renault is larger than those of other brands during the whole studied period.

Elasticity of the ratio $\frac{S_j}{S_{j'}}$ relative to $\check{X}_l$ (Table 10 in the Appendix A.4)

In Model A, the elasticity of a ratio $S_j/S_{j'}$ relative to $\check{X}_j$ is equal to 0.0267, whereas in Model B it can be smaller or larger according to the considered brands: the largest elasticity is for S_R/S_Z

[11] NB: here we take an example for an arbitrary share of 25% using the average elasticity. However, the only way to ensure that the sum of the modified shares $\sum_{m=1}^{D} S'_{mt}$ is equal to 1 is to use the corresponding elasticities calculated at the same time t, not the average elasticities.

ISBN: 978-84-947240-0-8

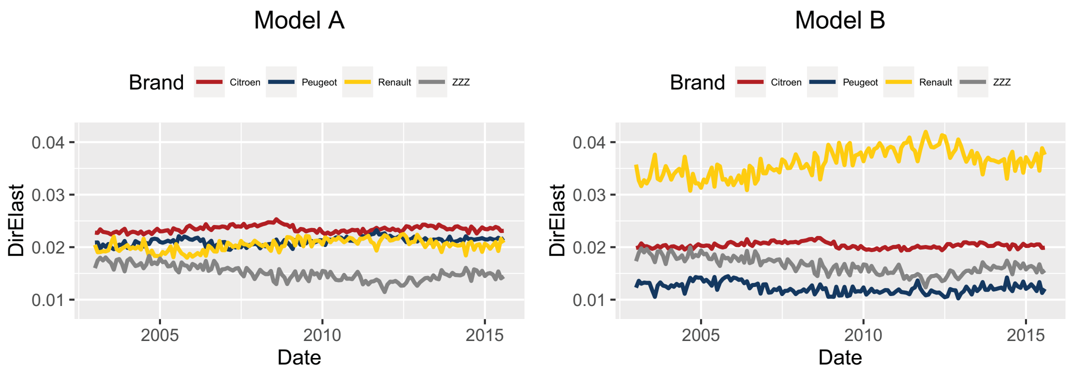

Figure 4: Direct elasticity of S_{jt} relative to $M_{j,t-1}$ across time

Table 8: Average elasticity of market-shares relative to media investments $\check{M}_{t-1}$

$e(S_{jt}, \check{M}_{l,t-1})$	Model A				Model B			
	$\check{M}_{C,t-1}$	$\check{M}_{P,t-1}$	$\check{M}_{R,t-1}$	$\check{M}_{Z,t-1}$	$\check{M}_{C,t-1}$	$\check{M}_{P,t-1}$	$\check{M}_{R,t-1}$	$\check{M}_{Z,t-1}$
$S_{Citroen,t}$	**0.0235**	-0.0056	-0.0063	-0.0116	**0.0204**	-0.0028	-0.0097	-0.0078
$S_{Peugeot,t}$	-0.0032	**0.0211**	-0.0063	-0.0116	-0.0054	**0.0099**	0.0119	-0.0163
$S_{Renault,t}$	-0.0032	-0.0056	**0.0204**	-0.0116	-0.0043	-0.0173	**0.0327**	-0.0111
$S_{Others,t}$	-0.0032	-0.0056	-0.0063	**0.0151**	-0.0008	0.0054	-0.0208	**0.0161**

C: Citroen; P: Peugeot; R: Renault; Z: Others.
Figures in bold: direct elasticities.

relative to $\check{X}_R$ which is equal to 0.0535. In general, ratios between the market-share of Renault and another brand are quite positively sensitive to media investments of Renault. For example, if the ratio S_R/S_Z is equal to $25/40 = 0.6250$ and Renault increases by 1% its media investments, then the ratio will increase to 0.6253. Let us remind that this measure does not depend on the considered period. This evolution is consistent with the fact that the market-share of Renault is very positively elastic and the market-share of "Others" is very negatively elastic to Renault media investments, as seen in Table 8.

Odds ratio of $\frac{S_j}{S_{j'}}$ to a change of $\check{X}_l$ (Table 11 in the Appendix A.4)
As expected, this measure is consistent with the previous one. In Model A, the odds ratio of any couple of brand market-shares $S_j/S_{j'}$ to a change of 10% of $\check{M}_{j,t-1}$ is equal to 1.0025, whereas it can reach 1.0054 in Model B for the ratio S_R/S_Z for a change of 10% in $\check{M}_{R,t-1}$. It means that if the ratio of market-shares of Renault over Others is equal to $25/40 = 0.6250$ and Renault decides to increase its media budget by 10%, then this ratio will increase to 0.6266 according to Model A and to 0.6284 according to Model B.

Elasticity of $\frac{S_j}{g(S_{-j})}$ relative to $\check{X}_l$ (Table 12 in the Appendix A.4)
As in Model A, no matter which transformation is used, the parameter b_1 will be the same, then we obtain that $e\left(\frac{S_{jt}}{g(S_{-jt})}, \frac{M_{j,t-1}}{g(M_{-j,t-1})}\right) = e\left(\frac{S_{jt}}{S_{j't}}, M_{j,t-1}\right) = e\left(\frac{S_{jt}}{S_{j't}}, \frac{M_{j,t-1}}{M_{j',t-1}}\right)$. Moreover, these elasticities are consistent with previous impact measures, and the largest one concerns the ratio $\frac{S_R}{g(S_{-R})}$ relatively to the ratio $\frac{M_R}{g(M_{-R})}$, which is equal to 0.0389%. For example, let us consider a situation where the market-shares are the following: $(S_C, S_P, S_R, S_Z)' = (13, 22, 25, 40)'$, inducing that $\frac{S_R}{g(S_{-R})} = 1.1095$. Then, if Renault increases its media investments by 1% of the geometric average of other brands media investments, we can expect its market-share to move from 110.95% to 110.99% of the geometric average market-share of others.

ISBN: 978-84-947240-0-8

4.4 Complexity and goodness-of-fit

We have seen that Model A and Model B can be used for the same type of application. Model B is more complex than Model A because it allows to have component-specific parameters for each explanatory variables along with cross-effects parameters. The number of parameters to fit of Model B can be a serious limitation when the number of components D and the number of explanatory compositions K increase. For example, in our application Model A involves 10 parameters whereas Model B involves 42.

However, Model B is also more flexible than Model A in the sense that it allows to have positive synergies (positive interactions) between some shares, whereas cross elasticities of Model A are necessarily negative[12]. For example, we see in Table 8 that when media investments of Citroen increase, it tends to benefit also to “Others”, and when media investments of Renault increase, it tends to benefit to Peugeot.

Is the complexity of Model B useful to explain brands market-shares of the B segment? To answer this question, let us look at cross-validated quality measures [13] (Table 9). Quality measures agree that Model B is much better than Model A to fit brands market-shares of the B segment of the French automobile market.

Table 9: Quality measures - Model A and Model B

	R_T^2	R_A^2	KL_C	$RMSE$
Model A	0.3039	0.2578	0.0386	0.0324
Model B	0.4532	0.2816	0.0399	0.0318

5 Conclusion

The focus of this paper is to present two types of compositional models for the case when the dependent variable and some of the explanatory variables are compositions, and to interpret them. A composition is a vector of shares called components (for example the brands market-shares in a given market), which are positive numbers and sum up to one. Compositional models are transformation models: they use a log-ratio transformation to transform components into coordinates in order to enhance the estimation. The difference between Model A and Model B is due to the model specification: in Model A, a single global coefficient is associated to an explanatory composition, whereas in Model B we assume that each component of the explanatory composition has a specific impact on each component of the dependent variable. Thus, in Model B, cross-effects between components are explicitly specified and can be positive, whereas in Model A they are implicit and negative by construction. Consequently, Model B is more flexible but also much more complex than Model A, and the number of parameters to fit can be a serious limitation to use it.

This paper presents a set of possible measures, mutually consistent, to interpret parameters of these two models: marginal effects, elasticities and odds ratios. The elasticity of a component relative to an explanatory variable is the relative variation of this component to a relative variation of the explanatory variable, ceteris paribus. This type of measure is totally adapted to enhance the interpretability of these models. However, this measure is observation dependent and we have to make sure that it is stable across observations to use it. Marginal effects are not well adapted to interpret this kind of models because they depend a lot on the considered observation. The other types of measures presented have the advantage to be observation independent, but they are more difficult to interpret in practical cases because they involve ratios.

[12] As long as the direct elasticity is positive (the cross elasticity is of opposite sign of the direct elasticity by construction).

[13] The out-of-sample computation process and the quality measures used are the same than in Morais et al. (2016).

ISBN: 978-84-947240-0-8

The two models are applied to the B segment of the French automobile market, for the purpose of measuring the impact of brand media investments on brands market-shares. Model B fits our data better than Model A according to several quality measures. In Model B, Renault is the brand which has the largest direct elasticity to media investments. The model shows interesting non-symmetric synergies between brands.

In a further work, it would be interesting to mix Model A and Model B in order to chose to put more or less flexibility on each explanatory variable. As compositions are observed across time, the potential autocorrelation of error terms has to be considered. Moreover, from a marketing point of view, it would be interesting to measure the impact of each channel (TV, radio, press, outdoor, internet, cinema) separately.

Acknowledgements

We thank BVA and the Marketing Direction of Renault for sharing valuable data with us, and for their support during the model specification and interpretation. This work was supported by the market research agency BVA and the French national research agency ANRT.

References

Chen, J., X. Zhang, and S. Li (2016). Multiple linear regression with compositional response and covariates. *Journal of Applied Statistics*, 1–16.

Hron, K., P. Filzmoser, and K. Thompson (2012). Linear regression with compositional explanatory variables. *Journal of Applied Statistics 39*(5), 1115–1128.

Morais, J., C. Thomas-Agnan, and M. Simioni (2016, 12). A tour of regression models for explaining shares. *TSE Working Paper* (16-742).

Pawlowsky-Glahn, V. and A. Buccianti (2011). *Compositional data analysis: Theory and applications*. John Wiley & Sons.

Pawlowsky-Glahn, V., J. J. Egozcue, and R. Tolosana-Delgado (2015). *Modeling and Analysis of Compositional Data*. John Wiley & Sons.

Van Den Boogaart, K. G. and R. Tolosana-Delgado (2013). *Analysing Compositional Data with R*. Springer.

Wang, H., L. Shangguan, J. Wu, and R. Guan (2013). Multiple linear regression modeling for compositional data. *Neurocomputing 122*, 490–500.

ISBN: 978-84-947240-0-8

A Appendix

A.1 Model A is a particular case of Model B

Let consider a Model B where $D_S = D_X = 3$, where the matrix of coefficients in the transformed space is equal to $\mathbf{B}^* = \begin{bmatrix} b^* & 0 \\ 0 & b^* \end{bmatrix}$, and where $\mathbf{V} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & 0 \end{bmatrix}$. Then, $\mathbf{B} = \mathbf{V}\mathbf{B}^*\mathbf{V}' = \frac{1}{3}b^* \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ such that the matrix $\mathbf{B}$ does verify the rows sum and columns sum equal to 0 requirement.

We can check that in this case we have $\mathbf{B} \boxdot \mathbf{X} = b \odot \mathbf{X}$:

$$\begin{aligned}
\mathbf{B} \boxdot \mathbf{X} &= \mathcal{C}(X_1^{\frac{2}{3}b} X_2^{-\frac{1}{3}b} X_3^{-\frac{1}{3}b} \ , \ X_1^{-\frac{1}{3}b} X_2^{\frac{2}{3}b} X_3^{-\frac{1}{3}b} \ , \ X_1^{-\frac{1}{3}b} X_2^{-\frac{1}{3}b} X_3^{\frac{2}{3}b})' \\
&= \mathcal{C}(X_1^b (X_1X_2X_3)^{-\frac{1}{3}b} \ , \ X_2^b (X_1X_2X_3)^{-\frac{1}{3}b} \ , \ X_3^b (X_1X_2X_3)^{-\frac{1}{3}b})' \\
&= \mathcal{C}(X_1^b \ , \ X_2^b \ , \ X_3^b)' = b \odot \mathbf{X}
\end{aligned}$$

Then, in this particular case, the Model B specification is equivalent to the Model A specification.

A.2 Marginal effect calculus

We are going to demonstrate how to compute marginal effects of the volume $\check{X}_{lt}$ on the dependent shares S_{jt}, and elasticities of S_{jt} relative to $\check{X}_{lt}$, using the transformed and the non-transformed models. The demonstration is made for Model B, with $D = 3$ components and an ILR transformation defined by the transformation matrix $\mathbf{V} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$. Let us remind that $\mathbf{X}^* = ilr(\mathbf{X}) = \mathbf{V}' \log(\mathbf{X})$, and $\mathbf{X} = ilr^{-1}(\mathbf{X}^*) = \mathcal{C}(\exp(\mathbf{V}\mathbf{X}^*))$.

We define the following transformations:

$$\begin{aligned}
T &: (\check{X}_1, \check{X}_2, \check{X}_3)' \to (\check{X}_1^*, \check{X}_2^*)' \\
F &: (\check{X}_1^*, \check{X}_2^*)' \to (\mathbb{E}S_1^*, \mathbb{E}S_2^*)' = (a_1^* + b_{11}^*\check{X}_1^* + b_{12}^*\check{X}_2^*, a_2^* + b_{21}^*\check{X}_1^* + b_{22}^*\check{X}_2^*)' \\
T^{-1} &: (\mathbb{E}S_1^*, \mathbb{E}S_2^*)' \to (\mathbb{E}^{\oplus}S_1, \mathbb{E}^{\oplus}S_2, \mathbb{E}^{\oplus}S_3)'
\end{aligned}$$

We are going to use the following property of Jacobian matrices: $J = J_{T^{-1}} J_F J_T$, implying that:

$$ME(\mathbb{E}^{\oplus}\mathbf{S}_t, \check{\mathbf{X}}_t) = \left[\frac{\partial \mathbb{E}^{\oplus} S_{it}}{\partial \check{X}_{jt}}\right]_{D,D} = \left[\frac{\partial \mathbb{E}^{\oplus} S_{it}}{\partial \mathbb{E}S_{jt}^*}\right]_{D,D-1} \left[\frac{\partial \mathbb{E}S_{it}^*}{\partial \check{X}_{jt}^*}\right]_{D-1,D-1} \left[\frac{\partial \check{X}_{it}^*}{\partial \check{X}_{jt}}\right]_{D-1,D}$$

and

$$E(\mathbb{E}^{\oplus}\mathbf{S}_t, \check{\mathbf{X}}_t) = \left[\frac{\partial \log \mathbb{E}^{\oplus} S_{it}}{\partial \log \check{X}_{jt}}\right]_{D,D} = \left[\frac{\mathbf{1}}{\mathbf{S_{it}}}\right] \boldsymbol{\odot} \left[\frac{\partial \mathbb{E}^{\oplus} S_{it}}{\partial \mathbb{E}S_{jt}^*}\right]_{D,D-1} \left[\frac{\partial \mathbb{E}S_{it}^*}{\partial \check{X}_{jt}^*}\right]_{D-1,D-1} \left[\frac{\partial \check{X}_{it}^*}{\partial \check{X}_{jt}}\right]_{D-1,D} \boldsymbol{\odot} [\mathbf{X_{jt}}]$$

where $\boldsymbol{\odot}$ denotes the Hadamard product here (term by term product)[14], $\left[\frac{\mathbf{1}}{\mathbf{S_{it}}}\right]$ is a $D \times D-1$ matrix with $1/S_{it}$ on the i^{th} row and $[\mathbf{X_{jt}}]$ is a $D-1, D$ matrix with X_{jt} on the j^{th} column.

[14] Note that $\boldsymbol{\odot}$ in bold denote the Hadamard product whereas $\odot$ denote the power transformation.

ISBN: 978-84-947240-0-8

The Jacobian of the model in coordinates J_F

$$J_F = \begin{bmatrix} \frac{\partial \mathbb{E}S_1^*}{\partial \check{X}_1^*} & \frac{\partial \mathbb{E}S_1^*}{\partial \check{X}_2^*} \\ \frac{\partial \mathbb{E}S_2^*}{\partial \check{X}_1^*} & \frac{\partial \mathbb{E}S_2^*}{\partial \check{X}_2^*} \end{bmatrix} = \begin{bmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{bmatrix} = \mathbf{B}^*$$

The Jacobian of the transformation J_T The ILR transformation is defined by:

$$(\check{X}_1^*, \check{X}_2^*)' = T(\check{X}_1, \check{X}_2, \check{X}_3)' = \left(\sqrt{\frac{2}{3}} \log \check{X}_1 - \frac{1}{\sqrt{6}} \log \check{X}_2 - \frac{1}{\sqrt{6}} \log \check{X}_3 \ , \ \frac{1}{\sqrt{2}} \log \check{X}_2 - \frac{1}{\sqrt{2}} \log \check{X}_3 \right)'$$

Then, $J_T = \begin{bmatrix} \frac{\partial \check{X}_1^*}{\partial \check{X}_1} & \frac{\partial \check{X}_1^*}{\partial \check{X}_2} & \frac{\partial \check{X}_1^*}{\partial \check{X}_3} \\ \frac{\partial \check{X}_2^*}{\partial \check{X}_1} & \frac{\partial \check{X}_2^*}{\partial \check{X}_2} & \frac{\partial \check{X}_2^*}{\partial \check{X}_3} \end{bmatrix} = \mathbf{V}' \odot \left[\frac{\mathbf{1}}{\mathbf{X_j}}\right] = \begin{bmatrix} \sqrt{\frac{2}{3}} \frac{1}{X_1} & -\frac{1}{\sqrt{6}} \frac{1}{X_2} & -\frac{1}{\sqrt{6}} \frac{1}{X_3} \\ 0 & \frac{1}{\sqrt{2}} \frac{1}{X_2} & -\frac{1}{\sqrt{2}} \frac{1}{X_3} \end{bmatrix}$
where $\left[\frac{\mathbf{1}}{\mathbf{X_j}}\right]$ is a $D-1, D$ matrix with $1/X_j$ on the j^{th} column.

The Jacobian of the inverse transformation $J_{T^{-1}}$

$$\begin{aligned} (\mathbb{E}^{\oplus} S_1, \mathbb{E}^{\oplus} S_2, \mathbb{E}^{\oplus} S_3)' &= T^{-1}(\mathbb{E}S_1^*, \mathbb{E}S_2^*)' = \mathcal{C}(\exp(\mathbf{V} \cdot \mathbb{E}\mathbf{S}^*)') \\ &= \mathcal{C}\left(\exp(\mathbb{E}S_1^*)^{\sqrt{\frac{2}{3}}}; \exp(\mathbb{E}S_1^*)^{-\frac{1}{\sqrt{6}}} \exp(\mathbb{E}S_2^*)^{\frac{1}{\sqrt{2}}}; \exp(\mathbb{E}S_1^*)^{-\frac{1}{\sqrt{6}}} \exp(\mathbb{E}S_2^*)^{-\frac{1}{\sqrt{2}}} \right)' \\ &= \left(\frac{u_1}{DEN}; \frac{u_2}{DEN}; \frac{u_3}{DEN} \right) \end{aligned}$$

where

$$\begin{aligned} u_1 &= \exp(\mathbb{E}S_1^*)^{\sqrt{\frac{2}{3}}} \\ u_2 &= \exp(\mathbb{E}S_1^*)^{-\frac{1}{\sqrt{6}}} \exp(\mathbb{E}S_2^*)^{\frac{1}{\sqrt{2}}} \\ u_3 &= \exp(\mathbb{E}S_1^*)^{-\frac{1}{\sqrt{6}}} \exp(\mathbb{E}S_2^*)^{-\frac{1}{\sqrt{2}}} \\ DEN &= u_1 + u_2 + u_3 \end{aligned}$$

In order to compute the matrix $J_{T^{-1}} = \begin{bmatrix} \frac{\partial \mathbb{E}^{\oplus} S_1}{\partial \mathbb{E}S_1^*} & \frac{\partial \mathbb{E}^{\oplus} S_1}{\partial \mathbb{E}S_2^*} \\ \frac{\partial \mathbb{E}^{\oplus} S_2}{\partial \mathbb{E}S_1^*} & \frac{\partial \mathbb{E}^{\oplus} S_2}{\partial \mathbb{E}S_2^*} \\ \frac{\partial \mathbb{E}^{\oplus} S_3}{\partial \mathbb{E}S_1^*} & \frac{\partial \mathbb{E}^{\oplus} S_3}{\partial \mathbb{E}S_2^*} \end{bmatrix}$, we need to compute the derivatives of the numerators of $\mathbb{E}^{\oplus}\mathbf{S}$: $\mathbf{u} = (u_1, u_2, u_3)'$ with respect to $\mathbb{E}\mathbf{S}^*$.

$$\left(\frac{\partial \mathbf{u}}{\partial \mathbb{E}\mathbf{S}^*} \right) = \mathbf{V} \odot \mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial \mathbb{E}S_1^*} = \sqrt{\frac{2}{3}} u_1 & \frac{\partial u_1}{\partial \mathbb{E}S_2^*} = 0 \\ \frac{\partial u_2}{\partial \mathbb{E}S_1^*} = -\frac{1}{\sqrt{6}} u_2 & \frac{\partial u_2}{\partial \mathbb{E}S_2^*} = \frac{1}{\sqrt{2}} u_2 \\ \frac{\partial u_3}{\partial \mathbb{E}S_1^*} = -\frac{1}{\sqrt{6}} u_3 & \frac{\partial u_3}{\partial \mathbb{E}S_2^*} = -\frac{1}{\sqrt{2}} u_3 \end{bmatrix}$$

Now we can compute the elements of $J_{T^{-1}}$. For example, the first element of this matrix is:

$$\frac{\partial \mathbb{E}^{\oplus} S_1}{\partial \mathbb{E}S_1^*} = \frac{DEN \sqrt{\frac{2}{3}} u_1 - u_1 [\sqrt{\frac{2}{3}} u_1 - \frac{1}{\sqrt{6}} u_2 - \frac{1}{\sqrt{6}} u_3]}{DEN^2} = \frac{\frac{3}{\sqrt{6}} u_1 (u_2 + u_3)}{DEN^2} = \frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 (1 - \mathbb{E}^{\oplus} S_1)$$

using the fact that $u_1/DEN = \mathbb{E}^{\oplus} S_1$ and $u_2 + u_3 = DEN - u_1$.

ISBN: 978-84-947240-0-8

Similar computations give the results for the whole matrix:

$$J_{T^{-1}} = \begin{bmatrix} \frac{\partial \mathbb{E}^{\oplus} S_1}{\partial \mathbb{E} S_1^*} & \frac{\partial \mathbb{E}^{\oplus} S_1}{\partial \mathbb{E} S_2^*} \\ \frac{\partial \mathbb{E}^{\oplus} S_2}{\partial \mathbb{E} S_1^*} & \frac{\partial \mathbb{E}^{\oplus} S_2}{\partial \mathbb{E} S_2^*} \\ \frac{\partial \mathbb{E}^{\oplus} S_3}{\partial \mathbb{E} S_1^*} & \frac{\partial \mathbb{E}^{\oplus} S_3}{\partial \mathbb{E} S_2^*} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 (1 - \mathbb{E}^{\oplus} S_1) & \frac{1}{\sqrt{2}} \mathbb{E}^{\oplus} S_1 (\mathbb{E}^{\oplus} S_3 - \mathbb{E}^{\oplus} S_2) \\ -\frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 \mathbb{E}^{\oplus} S_2 & \frac{1}{\sqrt{2}} \mathbb{E}^{\oplus} S_2 (\mathbb{E}^{\oplus} S_1 + 2\mathbb{E}^{\oplus} S_3) \\ -\frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 \mathbb{E}^{\oplus} S_3 & -\frac{1}{\sqrt{2}} \mathbb{E}^{\oplus} S_3 (\mathbb{E}^{\oplus} S_1 + 2\mathbb{E}^{\oplus} S_2) \end{bmatrix}$$

$$= [\mathbf{S_{it}}] \odot \begin{bmatrix} \frac{3}{\sqrt{6}} (1 - \mathbb{E}^{\oplus} S_1) & \frac{1}{\sqrt{2}} (\mathbb{E}^{\oplus} S_3 - \mathbb{E}^{\oplus} S_2) \\ -\frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 & \frac{1}{\sqrt{2}} (\mathbb{E}^{\oplus} S_1 + 2\mathbb{E}^{\oplus} S_3) \\ -\frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 & -\frac{1}{\sqrt{2}} (\mathbb{E}^{\oplus} S_1 + 2\mathbb{E}^{\oplus} S_2) \end{bmatrix} = [\mathbf{S_{it}}] \odot \mathbf{W}^*$$

The Jacobian of the model in the simplex J

$$J = J_{T^{-1}} J_F J_T = \begin{bmatrix} \frac{\partial S_1}{\partial \check{X}_1} & \frac{\partial S_1}{\partial \check{X}_2} & \frac{\partial S_1}{\partial \check{X}_3} \\ \frac{\partial S_2}{\partial \check{X}_1} & \frac{\partial S_2}{\partial \check{X}_2} & \frac{\partial S_2}{\partial \check{X}_3} \\ \frac{\partial S_3}{\partial \check{X}_1} & \frac{\partial S_3}{\partial \check{X}_2} & \frac{\partial S_3}{\partial \check{X}_3} \end{bmatrix}$$

$$= [\mathbf{S_{it}}] \odot \mathbf{W}^* \mathbf{B}^* \mathbf{V}' \odot \left[\mathbf{1}/\mathbf{\check{X}_j}\right] = [\mathbf{S_{it}}] \odot \mathbf{W}^* \mathbf{V}' \mathbf{B} \odot \left[\mathbf{1}/\mathbf{\check{X}_j}\right] = [\mathbf{S_{it}}] \odot \mathbf{WB} \odot \left[\mathbf{1}/\mathbf{\check{X}_j}\right]$$

$$= [\mathbf{S_{it}}] \odot \begin{bmatrix} \frac{3}{\sqrt{6}} (1 - \mathbb{E}^{\oplus} S_1) & \frac{1}{\sqrt{2}} (\mathbb{E}^{\oplus} S_3 - \mathbb{E}^{\oplus} S_2) \\ -\frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 & \frac{1}{\sqrt{2}} (\mathbb{E}^{\oplus} S_1 + 2\mathbb{E}^{\oplus} S_3) \\ -\frac{3}{\sqrt{6}} \mathbb{E}^{\oplus} S_1 & -\frac{1}{\sqrt{2}} (\mathbb{E}^{\oplus} S_1 + 2\mathbb{E}^{\oplus} S_2) \end{bmatrix} \begin{bmatrix} b_{11}^* & b_{12}^* \\ b_{21}^* & b_{22}^* \end{bmatrix} \begin{bmatrix} \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \odot \left[\mathbf{1}/\mathbf{\check{X}_j}\right]$$

$$= [\mathbf{S_{it}}] \odot \begin{bmatrix} 1 - S_1 & -S_2 & -S_3 \\ -S_1 & 1 - S_2 & -S_3 \\ -S_1 & -S_2 & 1 - S_3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \odot \left[\mathbf{1}/\mathbf{\check{X}_j}\right] = ME(\mathbb{E}^{\oplus} \mathbf{S}_t, \mathbf{\check{X}}_t)$$

$$\Leftrightarrow E(\mathbb{E}^{\oplus} \mathbf{S}_t, \mathbf{\check{X}}_t) = \left[\frac{\mathbf{1}}{\mathbf{S_{it}}}\right] \odot ME(\mathbb{E}^{\oplus} \mathbf{S}_t, \mathbf{\check{X}}_t) \odot \left[\mathbf{\check{X}_j}\right] = \mathbf{WB}$$

where $\mathbf{W}^* \mathbf{V}' = \mathbf{W}$ is a D, D matrix with $1 - S_i$ in the diagonal and $-S_i$ in the row i otherwise.

We then conclude that marginal effects and elasticities matrices are easy to compute using coefficients in the simplex or coefficients in the transformed space:

$$ME(\mathbb{E}^{\oplus} \mathbf{S}_t, \mathbf{\check{X}}_t) = [\mathbf{S_{it}}] \odot \mathbf{WB} \odot \left[\mathbf{1}/\mathbf{\check{X}_j}\right] = [\mathbf{S_{it}}] \odot \mathbf{WVB}^* \mathbf{V}' \odot \left[\mathbf{1}/\mathbf{\check{X}_j}\right]$$
$$E(\mathbb{E}^{\oplus} \mathbf{S}_t, \mathbf{\check{X}}_t) = \mathbf{WB} = \mathbf{WVB}^* \mathbf{V}'$$

A.3 Nullity of the sum of elasticities weighted by shares

We have to prove that $\sum_{m=1}^{D} e_{mlt} \mathbb{E}^{\oplus} S_{mt} = 0$. This is the necessary condition for new shares S'_{mt}, resulting from a change in X_{lt}, to sum up to one: $\sum_{m=1}^{D} S'_{mt} = 1 \Leftrightarrow \sum_{m=1}^{D} e_{mlt} \mathbb{E}^{\oplus} S_{mt} = 0$.
Proof:

$$\sum_{m=1}^{D} \mathbb{E}^{\oplus} S_{mt} = 1 \Leftrightarrow \sum_{m=1}^{D} \frac{\partial \mathbb{E}^{\oplus} S_{mt}}{\partial \log X_{lt}} = 0 \Leftrightarrow \sum_{m=1}^{D} \frac{\partial \mathbb{E}^{\oplus} S_{mt}}{\partial \log X_{lt}} \frac{1}{\mathbb{E}^{\oplus} S_{mt}} \mathbb{E}^{\oplus} S_{mt} = 0 \Leftrightarrow \sum_{m=1}^{D} e_{mlt} \mathbb{E}^{\oplus} S_{mt} = 0 \quad (18)$$

ISBN: 978-84-947240-0-8

A.4 Impact measures

Table 10: Elasticity of ratios of market-shares $\frac{S_{jt}}{S_{j't}}$ relative to media investments $\check{M}_{l,t-1}$

Model A		Model B							
	$\check{M}_{t-1}$		$\check{M}_{C,t-1}$		$\check{M}_{P,t-1}$		$\check{M}_{R,t-1}$		$\check{M}_{Z,t-1}$
$e\left(\frac{S_{jt}}{S_{j't}}, \check{M}_{j,t-1}\right)$	0.0267	$S_{C/P}$	0.0258	$S_{P/C}$	0.0127	$S_{R/C}$	0.0424	$S_{Z/C}$	0.0239
$e\left(\frac{S_{jt}}{S_{j't}}, \check{M}_{j',t-1}\right)$	-0.0267	$S_{C/R}$	0.0246	$S_{P/R}$	0.0272	$S_{R/P}$	0.0208	$S_{Z/P}$	0.0325
$e\left(\frac{S_{jt}}{S_{j't}}, \check{M}_{l,t-1}\right)^*$	0	$S_{C/Z}$	0.0211	$S_{P/Z}$	0.0044	$S_{R/Z}$	0.0535	$S_{Z/R}$	0.0273

*where $l \neq j, j'$ and $S_{C/Z}$ means $S_{Citroen,t}/S_{Others,t}$ for example.

Table 11: Odds ratios of market-shares for an increase of 10% in media investments $\check{M}_{l,t-1}$

Model A		Model B							
For $\Delta = 10\%$	$\check{M}_{t-1}$		$\check{M}_{C,t-1}$		$\check{M}_{P,t-1}$		$\check{M}_{R,t-1}$		$\check{M}_{Z,t-1}$
$\text{OR}\left(\frac{S_{jt}}{S_{j't}}, \check{M}_{j,t-1}, \Delta\right)$	1.0025	$S_{C/P}$	1.0025	$S_{P/C}$	1.0012	$S_{R/C}$	1.0045	$S_{Z/C}$	1.0022
$\text{OR}\left(\frac{S_{jt}}{S_{j't}}, \check{M}_{j',t-1}, \Delta\right)$	0.9975	$S_{C/R}$	1.0024	$S_{P/R}$	1.0030	$S_{R/P}$	1.0026	$S_{Z/P}$	1.0031
$\text{OR}\left(\frac{S_{jt}}{S_{j't}}, \check{M}_{l,t-1}, \Delta\right)^*$	0	$S_{C/Z}$	1.0020	$S_{P/Z}$	1.0007	$S_{R/Z}$	1.0054	$S_{Z/R}$	1.0028

*where $l \neq j, j'$ and $S_{C/Z}$ means $S_{Citroen,t}/S_{Others,t}$ for example.

Table 12: Elasticity of ratios $\frac{S_{jt}}{g(S_{-jt})}$ relative to $\check{M}_{l,t-1}$

Model A		Model B				
			$\check{M}_{C/g(-C)}$	$\check{M}_{P/g(-P)}$	$\check{M}_{R/g(-R)}$	$\check{M}_{Z/g(-Z)}$
$e\left(\frac{S_{jt}}{g(S_{-jt})}, \check{M}_{j,t-1}\right)$	0.0267	$S_{C/g(-C)}$	**0.0239**	-0.0022	-0.0176	-0.0040
		$S_{P/g(-P)}$	-0.0106	**0.0148**	0.0112	-0.0154
$e\left(\frac{S_{jt}}{g(S_{-jt})}, \check{M}_{l,t-1}\right)^*$	0	$S_{R/g(-R)}$	-0.0090	-0.0215	**0.0389**	-0.0085
		$S_{Z/g(-Z)}$	-0.0043	0.0089	-0.0324	**0.0279**

*where $l \neq j$.
$S_{C/g(-C)}$ means $\frac{S_{Ct}}{g(S_{-Ct})}$, where $g(S_{-Ct})$ is the geometric mean of others shares than Citroen.

ISBN: 978-84-947240-0-8